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Quasi-classical $\bar{\partial}$-dressing approach to the weakly dispersive KP hierarchy

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Abstract

The recently proposed quasi-classical $\bar{\partial}$-dressing method provides a systematic approach to studying the weakly dispersive limit of integrable systems. We apply the quasi-classical $\bar{\partial}$-dressing method to describe dispersive corrections of any order. We show how to calculate the $\bar{\partial}$ problems at any order for a rather general class of integrable systems, presenting explicit results for the KP hierarchy case. We demonstrate the stability of the method at each order. We construct an infinite set of commuting flows at first order which allows a description analogous to the zero-order (purely dispersionless) case, highlighting a Whitham-type structure. Obstacles for the construction of the higher order dispersive corrections are also discussed.

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1. Introduction

During the last decade great interest has been focused on the dispersionless limit of integrable nonlinear dispersive systems. They arise in various contexts of physics and mathematical physics: topological field theory and strings, matrix models, interface dynamics, nonlinear optics, conformal mapping theory [1–19]. Several methods and approaches have been used in order to study the features of this type of systems, such as the quasi-classical Lax pair representation with its close relationship with the Whitham universal hierarchy [4–7]; the hodograph transformations to calculate particular and interesting classes of exact solutions (rarefaction and shock waves) [9]; the quasi-classical version of the inverse scattering method allowing us to analyse various (1+1)-dimensional systems [10]. Moreover, the recent formulation of the quasi-classical $\bar{\partial}$-dressing method [20–22] introduces a more general and systematic approach to multidimensional integrable systems with weak dispersion, which preserves the power of the standard method. It allows us to build integrable systems and solutions simultaneously; that is a non-trivial fact, because of the difficulty of obtaining exact

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explicit solutions [23]. In addition, new interesting interrelations have been revealed, such as an intriguing connection with the quasi-conformal mapping theory, a strong similarity with the theory of semi-classical approximation to quantum mechanics [24] and geometric asymptotics methods to calculate wave corrections to geometrical optics [25]. While the method has been rather widely discussed for the purely dispersionless limit of some famous integrable hierarchies (KP, mKP, 2DTL) [20–22, 26, 27] the study of dispersive corrections is yet open. The latter is an interesting and challenging question in the context where one would like to be able to investigate the properties of the full dispersive system through an approximation theory.

A main goal of the present paper is to extend the quasi-classical $\bar{\partial}$-dressing method to higher order dispersive corrections, starting an investigation of the general properties at the dispersive orders of an integrable hierarchy. We will show that the quasi-classical $\bar{\partial}$-dressing method works as effectively as in the pure dispersionless case, and the first order admits a construction parallel to the leading order case. We present all calculations for the weakly dispersive KP hierarchy as an illustrative example. In this paper we do not consider the problem of construction of bounded solutions. So, we do not care about secular terms which usually appear in asymptotic expansion [28].

We briefly present the standard $\bar{\partial}$-method in section 2 and the quasi-classical $\bar{\partial}$-method in section 3. In section 4, we calculate the quasi-classical $\bar{\partial}$-problems at any order in the dispersive parameter, giving the explicit formulae at fourth order. In section 5 we recall the method at the first order [20], and prove a theorem allowing us to build an infinite set of flows reproducing the first corrections to the dKP equation. In section 6 we present the generalization of the $\bar{\partial}$-dressing method for higher orders. Section 7 contains some considerations about the possibility of describing the first order introducing a certain generalization of the Whitham hierarchy. Finally, we present some concluding remarks.

2. The standard $\bar{\partial}$-dressing method

The standard $\bar{\partial}$-dressing method is a powerful procedure allowing us to construct multidimensional integrable hierarchies of equations and their solutions. In this section we outline the main ideas of the method [29, 30]. The $\bar{\partial}$-dressing method is based on the nonlocal $\bar{\partial}$-problem

$$\frac{\partial \chi(z,\bar{z};t)}{\partial \bar{z}} = \int_{\mathbb{C}} \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, \chi(\mu,\bar{\mu};t)\, g(\mu,t)\, R_0(\mu,\bar{\mu};z,\bar{z})\, g^{-1}(z,t) \tag{1}$$

where z is the spectral parameter and $\bar{z}$ its complex conjugate, $t = (t_1, t_2, t_3, \ldots)$ the vector of the times, $\chi(z,\bar{z};t)$ a complex-valued function on the complex plane $\mathbb{C}$ and $R_0(\mu,\bar{\mu};z,\bar{z})$ is the $\bar{\partial}$-data.

The $\bar{\partial}$-equation (1) encodes all information about integrable hierarchies. It is assumed that the problem (1) is uniquely solvable. Usually, one considers its solutions with canonical normalization, i.e.

$$\chi \to 1 + \frac{\chi_1}{z} + \frac{\chi_2}{z^2} + \cdots \tag{2}$$

as $z \to \infty$.

The particular form of the function $g(z,t)$ sets an integrable hierarchy. Specifically, in order to describe the KP hierarchy one has to take

$$g(z,t) = \mathrm{e}^{S_0(z,t)} \qquad S_0(z,t) = \sum_{k=1}^{\infty} z^k t_k. \tag{3}$$

Following a standard procedure [30], it is possible to construct an infinite set of linear operators M_i, providing an infinite set of linear equations

$$M_i \chi = 0 \tag{4}$$

which are automatically compatible. Equations (4) provide us with the corresponding integrable hierarchy. For example, the pair of operators (Lax pair)

$$M_2 = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - u(x, y, t)$$
$$M_3 = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} - \frac{3}{2}u\frac{\partial}{\partial x} - \frac{3}{2}u_x - \frac{3}{4}\partial_x^{-1}u_y$$

with the notation

$$t_1 = x \qquad t_2 = y \qquad t_3 = t \qquad u_{t_i} = \frac{\partial u}{\partial t_i} \qquad \partial_x^{-1} = \int_{-\infty}^{x} \mathrm{d}x$$

provides us with the KP equation (sometimes called '*dispersionful*' KP)

$$\frac{\partial^2 u}{\partial t\, \partial x} = \frac{\partial}{\partial x}\left(\frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x}\right) + \frac{3}{4}\frac{\partial^2 u}{\partial y^2}. \tag{5}$$

The $\bar{\partial}$-dressing method allows us to construct a wide class of exact solutions of the KP equation and the whole KP hierarchy.

3. Quasi-classical $\bar{\partial}$-dressing method

3.1. Dispersionless limit

In order to illustrate the quasi-classical $\bar{\partial}$ formalism, we, following [20], introduce slow variables $t_i = T_i/\epsilon$, $\partial_{t_i} = \epsilon\partial_{T_i}$ and assume that $g(z, T; \epsilon) = \exp(S_0(z, T)/\epsilon)$, where ϵ is a small dispersive parameter. We are looking for solutions of the $\bar{\partial}$-problem in the form

$$\chi\left(z, \bar{z}; \frac{T}{\epsilon}\right) = \tilde{\chi}(z, \bar{z}; T; \epsilon) \exp\left(\frac{\tilde{S}(z, \bar{z}; T)}{\epsilon}\right) \tag{6}$$

where $\tilde{\chi}(z, \bar{z}; T; \epsilon) = \sum_{n=0}^{\infty} \varphi_n(z, \bar{z}; T)\epsilon^n$ is an asymptotic expansion. The function $\tilde{S}$ has the following expansion:

$$\tilde{S} \to \frac{\tilde{S}_1}{z} + \frac{\tilde{S}_2}{z^2} + \frac{\tilde{S}_3}{z^3} + \cdots \qquad z \to \infty. \tag{7}$$

Let us consider the $\bar{\partial}$-kernel of the following, quite general form:

$$R_0(\mu, \bar{\mu}; z, \bar{z}) = \frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}(-1)^k\, \epsilon^{k-1}\, \delta^{(k)}(\mu - z)\Gamma_k(z, \bar{z}). \tag{8}$$

The derivative δ-functions $\delta^{(k,n)}$ are defined, in the standard way, as the distributions such that

$$-\frac{1}{2\mathrm{i}}\int_{\mathbb{C}} \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, \delta^{(k,n)}(\mu - z) f(\mu, \bar{\mu}) = (-1)^{k+n}\, \frac{\partial^{k+n} f(z, \bar{z})}{\partial z^k \partial \bar{z}^n} \tag{9}$$

and $\delta^{(k)} := \delta^{(k,0)}$.

Introducing

$$S(z, \bar{z}; T) = S_0(z; T) + \tilde{S}(z, \bar{z}; T) \tag{10}$$

one has, in the KP case,

$$S = \sum_{k=1}^{\infty} z^k T_k + \sum_{j=1}^{\infty} \frac{\tilde{S}_j}{z^j} \qquad z \to \infty. \tag{11}$$

A straightforward calculation for the problem (1) with the kernel (8) gives the dispersionless (classical) limit of $\bar{\partial}$-problem

$$\frac{\partial S}{\partial \bar{z}} = W\left(z, \bar{z}, \frac{\partial S}{\partial z}\right) \tag{12}$$

where

$$W = \sum_{k=0}^{\infty} (-1)^k \left(\frac{\partial S}{\partial z}\right)^k \Gamma_k. \tag{13}$$

In this limit the role of the nonlocal linear $\bar{\partial}$-problem (1) is held by the nonlinear $\bar{\partial}$-equation (12), that is a local nonlinear equation of Hamilton–Jacobi type. The latter encodes all information about the integrable systems, as in the dispersionful case. Nevertheless, there are substantial technical differences in the dressing procedure. In particular, the quasi-classical $\bar{\partial}$-method is based crucially on the *Beltrami equation*'s properties. We present them in the next subsection.

3.2. The Beltrami equation (BE)

The BE has the form

$$\frac{\partial \Psi}{\partial \bar{z}} = \Omega(z, \bar{z}) \frac{\partial \Psi}{\partial z} \tag{14}$$

where $z \in \mathbb{C}$. Under certain conditions, it has the following properties (see e.g. [32]):

(i) If Ω satisfies $|\Omega| < k < 1$, then the only solution of BE such that $\frac{\partial \Psi}{\partial \bar{z}}$ is locally L^p for some $p > 2$, and such that Ψ vanishes at some point of the extended plane $\mathbb{C}^*$ is $\Psi \equiv 0$ (*Vekua's theorem*).

(ii) If $\Psi_1, \Psi_2, \ldots, \Psi_N$ are solutions of BE, a differentiable function $f(\Psi_1, \Psi_2, \ldots, \Psi_N)$ with arbitrary N is a solution of BE too.

(iii) The solutions of the BE under the previous conditions give quasi-conformal maps with the complex dilatation $\Omega(z, \bar{z})$ [33].

It is assumed, in all our further constructions, that these properties of BE are satisfied.

3.3. Zero order

The zero-order case has been widely discussed, for different integrable hierarchies in various papers [20–22, 26, 27]. Here, we outline the procedure for the KP hierarchy. The first crucial observation is that the symmetries of the $\bar{\partial}$-problem at leading order in ϵ, for any time T_j, i.e. $\delta S = \frac{\partial S}{\partial T_j} \delta T_j$, are given by the BE

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\partial S}{\partial T_j}\right) = W' \frac{\partial}{\partial z} \left(\frac{\partial S}{\partial T_j}\right). \tag{15}$$

We assume that the complex dilatation $W' = \frac{\partial W'(z, \bar{z}, \xi)}{\partial \xi}$ has good properties required for the application of the Vekua theorem. From (11) it follows that

$$\frac{\partial S}{\partial T_j} = z^j + \frac{1}{z} \frac{\partial \tilde{S}_1}{\partial T_j} + \frac{1}{z^2} \frac{\partial \tilde{S}_2}{\partial T_j} + \cdots \qquad z \to \infty. \tag{16}$$

Using the BE properties, one gets the following equations in time variables [20, 21],

$$\frac{\partial S}{\partial y} - \left(\frac{\partial S}{\partial x}\right)^2 - u_0(x, y, t) = 0 \tag{17}$$

$$\frac{\partial S}{\partial t} - \left(\frac{\partial S}{\partial x}\right)^3 - \frac{3}{2}u_0\frac{\partial S}{\partial x} - V_0(x, y, t) = 0 \tag{18}$$

where the notation

$$T_1 = x \qquad T_2 = y \qquad T_3 = t \tag{19}$$

is adopted and

$$u_0 = -2\frac{\partial \tilde{S}_1}{\partial x} \qquad \frac{\partial V_0}{\partial x} = \frac{3}{4}\frac{\partial u_0}{\partial y}. \tag{20}$$

Indeed, due to the property 2 of BE, the left-hand sides of equations (17) and (18) are solutions of BE and since they vanish at $z \to \infty$, then, by virtue of the *Vekua theorem*, they vanish identically on the whole complex plane. Equations (17) and (18) are automatically compatible and by expansion in $1/z$-power, according to the standard $\bar{\partial}$-dressing procedure, one obtains the well-known dispersionless KP (dKP) equation (or Zabolotskaya–Khokhlov equation)

$$\frac{\partial^2 u_0}{\partial t\, \partial x} = \frac{3}{2}\frac{\partial}{\partial x}\left(u_0\frac{\partial u_0}{\partial x}\right) + \frac{3}{4}\frac{\partial^2 u_0}{\partial y^2}. \tag{21}$$

Usually the dKP equation and its dispersive corrections are obtained, by a slow time limit, from the dispersionful KP equation (5), inserting the following asymptotic expansion in terms of a small dispersive parameter

$$u\left(\frac{T}{\epsilon}\right) = \sum_{n=0}^{\infty} u_n(T)\epsilon^n. \tag{22}$$

We stress that the formal expansion (22) implies, in general, the appearance of secular terms. Usually one imposes additional conditions on equations (17) and (18) to ensure the uniform validity of this expansion [28]. In this paper we do not care about boundedness of solutions (22) and, consequently, about secular terms.

Actually, as is well known, equations (17) and (18) are the first of an infinite set of equations in all time variables T_n, with $n \in \mathbb{N}\setminus\{0\}$, reproducing the whole KP hierarchy. Assuming

$$W(z, \bar{z}, T) = \theta(r - |z|)V\left(z, \bar{z}, \frac{\partial S}{\partial z}\right) \tag{23}$$

with $r > 0$, the function S is an analytic function outside the circle $\mathcal{D} = \{z \in \mathbb{C}||z| < r\}$. That is in agreement with the asymptotic behaviour (11). Finally the dKP hierarchy can be written in compact form as follows:

$$\frac{\partial S}{\partial T_n}(z, T) = \Omega_n(p(z, T), T) \qquad n \geqslant 1 \tag{24}$$

where $z \in \mathbb{C}\setminus\mathcal{D}$, and

$$p := \frac{\partial S}{\partial x}. \tag{25}$$

Taking into account that

$$p = z + \sum_{j=1}^{\infty} \frac{1}{z^j}\frac{\partial S_j}{\partial x} \tag{26}$$

$$\frac{\partial S}{\partial T_n} = z^n + O\left(\frac{1}{z}\right) \qquad z \to \infty \tag{27}$$

then

$$\Omega_n(p,T) - \left(\mathcal{Z}_0^n(p,T)\right)_+ = \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty \tag{28}$$

where $\mathcal{Z}_0$ denotes the expansion for z obtained by inversion of equation (26), and the symbol $(\cdot)_+$ means the polynomial part of the expansion. The left-hand sides of equations (28) are solutions of BE, so they vanish identically [22], i.e.

$$\Omega_n(p,T) = \left(\mathcal{Z}_0^n(p,T)\right)_+. \tag{29}$$

Then, Ω_n can be connected to a suitable expansion series in terms of the variable p. Assuming

$$\mathcal{Z}_0(p,T) = p + \sum_{j=1}^{\infty} \frac{a_j(T)}{p^j} \tag{30}$$

one has

$$\Omega_1 = p \qquad \Omega_2 = p^2 + 2a_1 \qquad \Omega_3 = p^3 + 3a_1 p + 3a_2 \tag{31}$$

that reproduces equations (17) and (18), by identifications

$$u_0 = 2a_1 \qquad V_0 = 3a_2. \tag{32}$$

The set of flows Ω_n represents a set of quasi-conformal maps for which the dispersionless integrable hierarchy describes a class of integrable deformations [21].

4. Quasi-classical $\bar{\partial}$-dressing method at any order

In this section we calculate the corrections at any order for the kernel (8),

$$R_0(\mu,\bar{\mu};z,\bar{z}) = \frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}(-1)^k \epsilon^{k-1}\,\delta^{(k)}(\mu - z)\Gamma_k(z,\bar{z}).$$

Let us observe that the independence of $\Gamma_{k,n}$ from the integration variables μ and $\bar{\mu}$ does not reduce the generality of the kernel (8). Indeed, by the delta function properties, it provides the same result, up to a redefinition, as for the kernels where $\Gamma_{k,n}$ depends on integration variables μ and $\bar{\mu}$.

Substituting expansion (6) into equation (1), one obtains, in a direct way, the quasi-classical $\bar{\partial}$-problem at any order. For this purpose, it is convenient to use the Faà de Bruno polynomials [31] defined as follows,

$$h_n[g(x)] = \partial_x h_{n-1}[g(x)] + h[g(x)]h_{n-1}[g(x)] \tag{33}$$

with $h_0[g(x)] = 1$ and $h_1[g(x)] = h[g(x)] = \partial_x g(x)$, $\partial_x \equiv \partial/\partial x$. Let us note that $h_n[g(x)] = h_n[g(x)]\left(\partial_x g, \partial_x^{(2)} g, \ldots, \partial_x^{(n)} g\right)$, in other words, it depends on the derivatives of g upto nth order.

In our case we have

$$h_l[\tilde{k}\log S(z,\bar{z})] \tag{34}$$

for some $\tilde{k} \in \mathbb{N}$. Observing that

$$h_l[\tilde{k}\log S(z,\bar{z})] = \sum_{l'=1}^{l} C_{l,l'} S^{-l'} \tag{35}$$

we denote

$$\mathcal{H}_{s,l} = \frac{C_{s,l}}{l!\binom{\tilde{k}}{l}} \tag{36}$$

where $\tilde{k}$ is an arbitrary integer larger than l. We note that the quantity (36) does not depend on $\tilde{k}$. Now, we introduce the operator $\hat{W}_{p,q}$

$$\hat{W}_{p,q} = W_{p,q}\partial_z^p \qquad W_{p,q} = \sum_{k=0}^{\infty}(-1)^k\binom{k}{p}\mathcal{H}_{k-p,k-q}\Gamma_{k,n}.$$

In particular, we have

$$W = \hat{W}_{0,0} = \sum_{k=0}^{\infty}(-1)^k\mathcal{H}_{k,k}\Gamma_{k,n} \qquad \mathcal{H}_{k,k} = \left(\frac{\partial S}{\partial z}\right)^k.$$

So, we have all the ingredients to write the quasi-classical $\bar\partial$-problem at nth order:

$$\frac{\partial S}{\partial\bar{z}}(z,\bar{z}) = W\left(z,\bar{z},\frac{\partial S}{\partial z}\right) \tag{37}$$

$$\frac{\partial\varphi_0}{\partial\bar{z}} = W'\frac{\partial\varphi_0}{\partial z} + \frac{1}{2}W''\frac{\partial^2 S}{\partial z^2}\varphi_0 \tag{38}$$

$$\cdots$$

$$\frac{\partial\varphi_n}{\partial\bar{z}} = W'\frac{\partial\varphi_n}{\partial z} + \frac{1}{2}W''\frac{\partial^2 S}{\partial z^2}\varphi_n + \sum_{q=1}^{n+1}\hat{W}_{p,q}\varphi_{n-q+1} \tag{39}$$

where

$$W^{(i)} = \frac{\partial^i}{\partial\xi^i}W(z,\bar{z},\xi). \tag{40}$$

It is interesting to observe that equation (37) is a Hamilton–Jacobi-type equation, the first-order correction is a homogeneous linear equation, while the next orders are linear equations, but nonhomogeneous ones. In agreement with the normalization condition (2) we have

$$\begin{aligned}
&\varphi_0 \to 1 + \frac{\varphi_{0,1}}{z} + \frac{\varphi_{0,2}}{z^2} + \frac{\varphi_{0,3}}{z^3} + \cdots\\
&\varphi_1 \to \frac{\varphi_{1,1}}{z} + \frac{\varphi_{1,2}}{z^2} + \frac{\varphi_{1,3}}{z^3} + \cdots\\
&\cdots\\
&\varphi_n \to \frac{\varphi_{n,1}}{z} + \frac{\varphi_{n,2}}{z^2} + \frac{\varphi_{n,3}}{z^3} + \cdots \qquad n\in\mathbb{N}\backslash\{0\}.
\end{aligned}$$

The explicit problem at fourth order, i.e. $n = 3$, is

$$\frac{\partial S}{\partial\bar{z}} = W\left(z,\bar{z},\frac{\partial S}{\partial z}\right) \tag{41}$$

$$D_0\varphi_0 = 0 \tag{42}$$

$$D_0\varphi_1 = D_1\varphi_0 \tag{43}$$

$$D_0\varphi_2 = D_1\varphi_1 + D_2\varphi_0 \tag{44}$$

$$D_0\varphi_3 = D_1\varphi_2 + D_2\varphi_1 + D_3\varphi_0 \tag{45}$$

where

$$D_0 = \frac{\partial}{\partial \bar{z}} - W' \frac{\partial}{\partial z} - \frac{1}{2} W'' \frac{\partial^2 S}{\partial z^2}$$

$$D_1 = \frac{1}{2} W'' \frac{\partial^2}{\partial z^2} + \frac{1}{2} W''' \frac{\partial^2 S}{\partial z^2} \frac{\partial}{\partial z} + \frac{1}{6} W''' \frac{\partial^3 S}{\partial z^3} + \frac{1}{8} W^{(4)} \left(\frac{\partial^2 S}{\partial z^2}\right)^2$$

$$D_2 = \frac{1}{6} W''' \frac{\partial^3}{\partial z^3} + \frac{1}{4} W^{(4)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^2}{\partial z^2} + \left(\frac{1}{6} W^{(4)} \frac{\partial^3 S}{\partial z^3} + \frac{1}{8} W^{(5)} \left(\frac{\partial^2 S}{\partial z^2}\right)^2\right) \frac{\partial}{\partial z} + \frac{1}{24} W^{(4)} \frac{\partial^4 S}{\partial z^4} + \frac{1}{12} W^{(5)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^3 S}{\partial z^3} + \frac{1}{48} W^{(6)} \left(\frac{\partial^2 S}{\partial z^2}\right)^3$$

$$D_3 = \frac{1}{24} W^{(4)} \frac{\partial^4}{\partial z^4} + \frac{1}{12} W^{(5)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^3}{\partial z^3} + \left(\frac{1}{12} W^{(5)} \frac{\partial^3 S}{\partial z^3} + \frac{1}{16} W^{(6)} \left(\frac{\partial^2 S}{\partial z^2}\right)^2\right) \frac{\partial^2}{\partial z^2} + \left(\frac{1}{24} W^{(5)} \frac{\partial^4 S}{\partial z^4} + \frac{1}{12} W^{(6)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^3 S}{\partial z^3} + \frac{1}{48} W^{(7)} \left(\frac{\partial^2 S}{\partial z^2}\right)^3\right) \frac{\partial}{\partial z} + \frac{1}{120} W^{(5)} \frac{\partial^5 S}{\partial z^5} + \frac{1}{72} W^{(6)} \left(\frac{\partial^3 S}{\partial z^3}\right)^2 + \frac{1}{48} W^{(6)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^4 S}{\partial z^4} + \frac{1}{48} W^{(7)} \left(\frac{\partial^2 S}{\partial z^2}\right)^2 \frac{\partial^3 S}{\partial z^3} + \frac{1}{384} W^{(8)} \left(\frac{\partial^2 S}{\partial z^2}\right)^4.$$

We stress the strong analogy between equations (41)–(45) and the Hamilton–Jacobi and higher transport equations which arise in the semi-classical approximation [24] and theory of geometric asymptotics [25]. Anyway, the main difference with the cited approaches is that in our case the role of the phase function is played by the function S that is a complex-valued one rather than a real-valued one.

5. First-order contribution

The BE's properties are very useful for the study of higher order corrections too [20]. This analysis has been initiated in the paper [20]. Some results of [20] are presented here for completeness. Fixing an arbitrary solution $S(z, \bar{z}, T)$ of equation (41), let φ_0 and $L\varphi_0$ be two solutions of equation (42), where L is a suitable linear operator depending on time variables. The ratio $L\varphi_0/\varphi_0$ satisfies the BE

$$\frac{\partial}{\partial \bar{z}} \left(\frac{L\varphi_0}{\varphi_0}\right) = W' \frac{\partial}{\partial z} \left(\frac{L\varphi_0}{\varphi_0}\right). \tag{46}$$

Then, choosing L such that

$$L\varphi_0 \to 0 \qquad z \to \infty \tag{47}$$

as a result of the Vekua theorem, one obtains

$$L\varphi_0 = 0 \qquad \forall z \in \mathbb{C}. \tag{48}$$

In particular, for the first equation of the KP hierarchy we have

$$L_1\varphi_0 \equiv \left(\frac{\partial}{\partial y} - 2\frac{\partial S}{\partial x} \frac{\partial}{\partial x} - \frac{\partial^2 S}{\partial x^2} - u_1(x, y, t)\right) \varphi_0 = 0 \tag{49}$$

$$L_2\varphi_0 \equiv \left[\frac{\partial}{\partial t} - \left(3\left(\frac{\partial S}{\partial x}\right)^2 + \frac{3}{2} u_0\right) \frac{\partial}{\partial x} - 3\frac{\partial S}{\partial x} \frac{\partial^2 S}{\partial x^2} - \frac{3}{2} \frac{\partial S}{\partial x} u_1 - \frac{3}{4} \frac{\partial u_0}{\partial x} - V_1\right] \varphi_0 = 0 \tag{50}$$

where

$$u_1 = -2\frac{\partial \varphi_{0,1}}{\partial x} \qquad \frac{\partial V_1}{\partial x} = \frac{3}{4}\frac{\partial u_1}{\partial y} \tag{51}$$

are extracted from the condition (47) and u_0 is an arbitrary solution of equation (21).

Setting to zero the $1/z$-power expansion's coefficients in (49) and (50), we find the first dispersive correction to the dKP equation

$$\frac{\partial^2 u_1}{\partial t\, \partial x} = \frac{3}{2}\frac{\partial^2}{\partial x^2}(u_0 u_1) + \frac{3}{4}\frac{\partial^2 u_1}{\partial y^2}. \tag{52}$$

Let us note that u_1 satisfies the equation defining the symmetries of the dKP equation (21). Indeed, considering equation (21) for $u_0 + \delta u_0$, one, obviously, obtains the equation

$$\frac{\partial^2(\delta u)}{\partial t \partial x} = \mathcal{S}_0(\delta u) \tag{53}$$

where

$$\mathcal{S}_0(\cdot) = \frac{3}{2}\frac{\partial^2}{\partial x^2}(u_0 \cdot) + \frac{3}{4}\frac{\partial^2}{\partial y^2}(\cdot) \tag{54}$$

that coincide with (52).

Now we will present some new results concerning the first-order corrections.

Theorem 1. *Let A_n and C_n be the differentiable functions defined by*

$$A_n = A_n(p, T) = \left(\frac{\mathrm{d}\left(\mathcal{Z}_0^n\right)}{\mathrm{d}p}\right)_+ \qquad C_n = C_n(p, \partial_x p, T) = \frac{1}{2}\frac{\mathrm{d}A_n}{\mathrm{d}x}(p, T)$$

where $\mathcal{Z}_0$ is given by (30), $B_n = B_n(p, T)$ an arbitrary differentiable function, φ_0 some solution of equation (42) and the linear operator $L^{(n)}$ is given by

$$L^{(n)} = \frac{\partial}{\partial T_n} - A_n\frac{\partial}{\partial x} - B_n - C_n. \tag{55}$$

Then $L^{(n)}\varphi_0$ is also the solution of equation (42).

Proof. By the use of equation (41) and $p = \frac{\partial S}{\partial x}(z, T)$ it can immediately be verified that

$$\frac{\partial A_n}{\partial \bar{z}} = W'\frac{\partial A_n}{\partial p}\frac{\partial}{\partial z}\frac{\partial S}{\partial x} = W'\frac{\partial A_n}{\partial z} \tag{56}$$

$$\frac{\partial B_n}{\partial \bar{z}} = W'\frac{\partial B_n}{\partial p}\frac{\partial}{\partial z}\frac{\partial S}{\partial x} = W'\frac{\partial B_n}{\partial z} \tag{57}$$

$$\frac{\partial C_n}{\partial \bar{z}} = W'\frac{\partial C_n}{\partial p}\frac{\partial}{\partial z}\frac{\partial S}{\partial x} + W'\frac{\partial C_n}{\partial(\partial_x p)}\frac{\partial^2 S}{\partial x^2} + W''\frac{\partial C_n}{\partial(\partial_x p)}\left(\frac{\partial}{\partial z}\frac{\partial S}{\partial x}\right)^2. \tag{58}$$

Differentiating equation (24) with respect to z, we get

$$\frac{\partial}{\partial z}\left(\frac{\partial S}{\partial T_n}\right) = A_n\frac{\partial}{\partial z}\left(\frac{\partial S}{\partial x}\right). \tag{59}$$

Moreover, the definition of the function C_n

$$C_n = \frac{1}{2}\frac{\mathrm{d}A_n}{\mathrm{d}x} = \frac{1}{2}\frac{\partial A_n}{\partial p}\frac{\partial p}{\partial x} + \frac{1}{2}\frac{\partial A_n}{\partial x} \tag{60}$$

implies that

$$\frac{\partial C_n}{\partial\,(\partial_x p)} = \frac{1}{2}\frac{\partial A_n}{\partial p}. \tag{61}$$

Calculating $D_0(L^{(n)}\varphi_0)$, where D_0 is given in equation (46), and exploiting equations (41), (42) and the set of equalities (56)–(59), (61), one finds

$$D_0(L^{(n)}\varphi_0) = 0. \tag{62}$$

This completes the proof. □

Based on theorem 1, we choose, in particular,

$$B_n = \left(\eta(p,T)\frac{\mathrm{d}\left(\mathcal{Z}_0^n\right)}{\mathrm{d}p}\right)_+ \tag{63}$$

where η is a series defined by

$$\eta(p,T) = \sum_{j=1}^{\infty}\frac{b_j(T)}{p^j}. \tag{64}$$

A choice of the coefficients in equations (30) and (64) in such a way that $L\varphi_0 \to 0$ for $z \to \infty$, together with the previous arguments on the BE, gives us the following set of linear problems in time variables:

$$L^{(n)}\varphi_0 = 0. \tag{65}$$

These equations can be rearranged by analogy with the leading order case, in the form

$$\frac{\partial \log\varphi_0}{\partial T_n}(z,T) = \Lambda_n(q(z,T), p(z,T), \partial_x p(z,T), T) \qquad z \in \mathbb{C}\backslash\mathcal{D} \tag{66}$$

where

$$\begin{aligned}
&\Lambda_n = \left(\mathcal{L}_1\frac{\mathrm{d}\left(\mathcal{Z}_0^n\right)}{\mathrm{d}p} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\left(\mathcal{Z}_0^n\right)}{\mathrm{d}p}\right)_+ \\
&\mathcal{L}_1 := q + \eta(p,T) \\
&q := \frac{\partial\log\varphi_0}{\partial x}.
\end{aligned} \tag{67}$$

Taking $n = 1, 2, 3$

$$\begin{aligned}
&\Lambda_1 = q \\
&\Lambda_2 = 2pq + 2b_1 + \partial_x p \\
&\Lambda_3 = 3p^2 q + 3a_1 q + 3b_1 p + 3p\partial_x p + \tfrac{3}{2}\partial_x a_1 + 3b_2
\end{aligned}$$

we reproduce equations (49) and (50) and, consequently, the first-order dispersive correction to the dKP equation by the identifications

$$u_0 = 2a_1 \qquad V_0 = 3a_2 \qquad u_1 = 2b_1 \qquad V_1 = 3b_2.$$

6. Higher order contributions

Unlike the first order, the higher order corrections are characterized by nonhomogeneous equations. Here we will show how the $\bar\partial$-dressing procedure works in these cases, discussing explicitly the KP equation.

Let us consider a set of functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ satisfying the $\bar\partial$ problem at $(n+1)$th order and $n+1$ linear operators in time variables $K^{(0)}, K^{(1)}, \ldots, K^{(n)}$ such that the quantity $\sum_{m=0}^{n} K^{(m)}\varphi_{n-m}$ satisfies equation (42). Then the ratio

$$\frac{\sum_{m=0}^{n} K^{(m)}\varphi_{n-m}}{\varphi_0} \tag{68}$$

is a solution of BE with complex dilatation W'.

Using the same arguments as in the first-order case, choosing $K^{(j)}$, $j = 0, \dots, n$, in such a way that $\sum_{m=0}^{n} K^{(m)}\varphi_{n-m} \to 0$ for $z \to \infty$, we get the linear equations

$$\sum_{m=0}^{n} K^{(m)}\varphi_{n-m} = 0 \qquad \forall z \in \mathbb{C}. \tag{69}$$

For instance, the second-order dispersive corrections to dKP equation are associated with the following pair,

$$K_1^{(0)}\varphi_1 + K_1^{(1)}\varphi_0 = 0 \tag{70}$$

$$K_2^{(0)}\varphi_1 + K_2^{(1)}\varphi_0 = 0 \tag{71}$$

where $K_1^{(0)} = L_1$ and $K_2^{(0)} = L_2$ are given by (49) and (50),

$$K_1^{(1)} = -\frac{\partial^2}{\partial x^2} - u_2 \tag{72}$$

$$K_2^{(1)} = -3\frac{\partial S}{\partial x}\frac{\partial^2}{\partial x^2} - \left(3\frac{\partial^2 S}{\partial x^2} + \frac{3}{2}u_1\right)\frac{\partial}{\partial x} - \frac{\partial^3 S}{\partial x^3} - \frac{3}{2}\frac{\partial S}{\partial x}u_2 - \frac{3}{4}\frac{\partial u_1}{\partial x} - V_2 \tag{73}$$

and

$$u_2 = -2\frac{\partial \varphi_{1,1}}{\partial x} \qquad \frac{\partial V_2}{\partial x} = \frac{3}{4}\frac{\partial u_2}{\partial y} \tag{74}$$

are deduced similarly to (51).

Equations (70) and (71) are compatible if and only if u_2 satisfies the second-order dispersive correction to dKP equation

$$\frac{\partial^2 u_2}{\partial t\, \partial x} = \frac{3}{2}\frac{\partial^2}{\partial x^2}(u_0 u_2) + \frac{3}{4}\frac{\partial^2 u_2}{\partial y^2} + \frac{3}{4}\frac{\partial^2 u_1^2}{\partial x^2} + \frac{1}{4}\frac{\partial^4 u_0}{\partial x^4}. \tag{75}$$

The third-order correction corresponds to the compatibility condition for the pair of the following linear problems

$$K_1^{(0)}\varphi_2 + K_1^{(1)}\varphi_1 + K_1^{(2)}\varphi_0 = 0 \tag{76}$$

$$K_2^{(0)}\varphi_2 + K_2^{(1)}\varphi_1 + K_2^{(2)}\varphi_0 = 0 \tag{77}$$

where

$$K_1^{(2)} = -u_3 \tag{78}$$

$$K_2^{(2)} = -\frac{\partial^3}{\partial x^3} - \frac{3}{2}u_3\frac{\partial S}{\partial x} - \frac{3}{2}u_2\frac{\partial}{\partial x} - \frac{3}{4}\frac{\partial u_2}{\partial x} - V_3 \tag{79}$$

with the definitions

$$u_3 = -2\frac{\partial \varphi_{2,1}}{\partial x} \qquad \frac{\partial V_3}{\partial x} = \frac{3}{4}\frac{\partial u_3}{\partial y}. \tag{80}$$

The equation for u_3 is of the form

$$\frac{\partial^2 u_3}{\partial t\, \partial x} = \frac{3}{2}\frac{\partial^2}{\partial x^2}(u_0 u_3) + \frac{3}{2}\frac{\partial^2}{\partial x^2}(u_1 u_2) + \frac{3}{4}\frac{\partial^2 u_3}{\partial y^2} + \frac{1}{4}\frac{\partial^4 u_1}{\partial x^4}. \tag{81}$$

A simple observation from equations (70) and (71) allows us to rewrite them in a compact form, suggesting the generalization to any order. In order to realize that, we introduce the following operator:

$$\Delta^k[S;\theta] = \begin{cases} \mathrm{e}^{-\theta S}\dfrac{\partial^k}{\partial x^k}\mathrm{e}^{\theta S} & \text{if} \quad k \geqslant 0 \\ 0 & \text{if} \quad k < 0. \end{cases} \tag{82}$$

Noting that for $k \geqslant 0$

$$\frac{\mathrm{d}^r}{\mathrm{d}\theta^r}\Delta^k[S;\theta] = \left[\cdots\left[\frac{\partial^k}{\partial x^k}, S\right], \ldots, S\right] \qquad r \text{ brackets} \tag{83}$$

the transport equations can be written as follows

$$\frac{\partial \varphi_n}{\partial T_k} = \sum_{r=0}^{k-1}\frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}\theta^r}\Delta^k[S;0]\varphi_{n-k+r+1} + \frac{k}{2}\sum_{j=0}^{n}\sum_{r=0}^{k-2} u_{j+r}\frac{\mathrm{d}^r}{\mathrm{d}\theta^r}\Delta^{k-2}[S;0]\varphi_{n-j}$$
$$+ \sum_{j=0}^{n}\left(\frac{3}{4}\frac{\partial u_j}{\partial x} + V_{j+1}\right)\Delta^{k-3}[S;0]\varphi_{n-j} \tag{84}$$

for $k = 1, 2, 3$.

It would be nice to get a similar formula for higher times too.

Finally, as we noted above, a solution u_1 of the first dispersive equation is defined up to a symmetry of the dKP equation. Using this freedom one can fix a 'gauge' putting $u_1 = 0$. Equations for higher corrections imply that the gauge $u_n = 0$ for each odd n is an admissible one. In such a gauge the even order corrective equations take the form

$$\frac{\partial^2 (u_n)}{\partial t \partial x} = \mathcal{S}_0 (u_n) + f_n \qquad n \text{ even}. \tag{85}$$

That is a nonhomogeneous equation with the corresponding homogeneous one given by the symmetry equation. By virtue of the Fredholm theorem, the nonhomogeneous term must be orthogonal to the solutions $\delta^* u$ of the adjoint homogeneous equation, i.e.

$$\int \delta^* u(P) f_n(P)\,\mathrm{d}P = 0 \qquad P = (x, y, t, \ldots). \tag{86}$$

One can check that for regular solutions of the KP equation, condition (86) is satisfied. A quite different situation takes place for the singular sector of the dKP equation. For instance, for breaking wave solutions for which $\delta u_0 \equiv \frac{\partial u_0}{\partial x} \to \infty$, condition (86) breaks. So, there are obstacles to constructing the higher order quasi-classical corrections for the dKP equation originated from its own singular sector. Such an obstacle is analogous to a typical one appearing in the construction of global asymptotics in the semi-classical approximation to the quantum mechanics [24] and in the study of the wave corrections to geometrical optics [25], because of the existence of caustics. In the dispersionless KdV case this kind of problem induces a stratification of the affine space of times providing a classification of the singularities [34]. The problem of obstacles for the construction of the higher order corrections to the dKP equation will be considered elsewhere.

7. A Whitham-type structure

It is well known that the dispersionless limit of some integrable hierarchies admits a symplectic structure [4–7]. In particular, one can see that starting with the function $S(z, T)$ outside the circle $\mathcal{D}$ it is possible to introduce a 2-form closed ω_0 in terms of the p-variable such that

$$\omega_0 = \mathrm{d}\Omega_n (p, T) \wedge \mathrm{d}T_n = \mathrm{d}\mathcal{Z}_0 (p, T) \wedge \mathrm{d}\tilde{\mathcal{M}}_0(p, T) \tag{87}$$

where the sum over index n is assumed. Here the last member contains the pair of Darboux coordinates $\mathcal{Z}_0$ and $\tilde{\mathcal{M}}_0(p, T) = \mathcal{M}_0(\mathcal{Z}_0(p, T), T) = \mathcal{M}_0(z, T)$ and the function

$$\mathcal{M}_0(z, T) := \frac{\partial S}{\partial z}(z, T) \tag{88}$$

is usually called Orlov's function [35].

From (87), it follows that

$$1 = \{\mathcal{Z}_0, \tilde{\mathcal{M}}_0\}_{p,x} \tag{89}$$

$$\frac{\partial \mathcal{Z}_0}{\partial T_n} = \{\Omega_n, \mathcal{Z}_0\}_{p,x} \tag{90}$$

$$\frac{\partial \tilde{\mathcal{M}}_0}{\partial T_n} = \{\Omega_n, \tilde{\mathcal{M}}_0\}_{p,x} \tag{91}$$

where the Poisson bracket is defined as follows:

$$\{f(\alpha,\beta), g(\alpha,\beta)\}_{\alpha,\beta} = \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial \beta} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial \beta}. \tag{92}$$

Equation (89) is usually referred to as the *string equation* [6] and equation (90) is the dispersionless Lax pair. Moreover, equation (87) implies the zero curvature condition

$$\omega_0 \wedge \omega_0 = 0. \tag{93}$$

That is another way to write the compatibility condition between equations (90) or (91); indeed, expanding the total differentials, one obtains from equation (93) the set of equations

$$\frac{\partial \Omega_n}{\partial T_m} - \frac{\partial \Omega_m}{\partial T_n} + \{\Omega_n, \Omega_m\}_{p,x} = 0 \tag{94}$$

that are the universal Whitham hierarchy equations [4]. In particular, for $n = 2$ and $m = 3$ one obtains equation (21).

The construction of the infinite set of flows in equation (66) allows us to describe the first correction in a way analogous to the zero curvature condition (93). Let us start with the total differential of $\log \varphi_0$

$$\mathrm{d}\log\varphi_0(z,T) = \Lambda_n(q(z,T), p(z,T), \partial_x p(z,T), T)\,\mathrm{d}T_n + \mathcal{M}_1(z,T)\,\mathrm{d}z. \tag{95}$$

The function

$$\mathcal{M}_1(z,T) = \frac{\partial \log\varphi_0}{\partial z}(z,T) \tag{96}$$

is an analogue of the Orlov function. Differentiating equation (95), one can introduce the 2-form

$$\omega_1 := \mathrm{d}\Lambda_n(q(z,T), p(z,T), \partial_x p(z,T), T) \wedge \mathrm{d}T_n = \mathrm{d}z \wedge \mathrm{d}\mathcal{M}_1(z,T). \tag{97}$$

Equation (87) allows us to identify a Whitham-type structure defined by the equation

$$\omega_1 \wedge \omega_1 = 0 \tag{98}$$

which can be given explicitly

$$\begin{aligned}\frac{\partial \Lambda_n}{\partial T_m} - \frac{\partial \Lambda_m}{\partial T_n} + \{\Lambda_n, \Lambda_m\}_{q,x} + \{\Lambda_n, \Lambda_m\}_{q,p}\frac{\partial p}{\partial x} + \{\Lambda_n, \Lambda_m\}_{q,\partial_x p}\frac{\partial^2 p}{\partial x^2}&\\ = G_n(\Lambda_m) - G_m(\Lambda_n)&\end{aligned} \tag{99}$$

where G_n is defined as

$$G_n(f) = \left(\frac{\mathrm{d}\Omega_n}{\mathrm{d}x}\frac{\partial}{\partial p} + \frac{\mathrm{d}^2\Omega_n}{\mathrm{d}x^2}\frac{\partial}{\partial(\partial_x p)}\right) f. \tag{100}$$

Let us note that for $n = 2$ and $m = 3$, equation (99) gives, of course, equation (52).

8. Concluding remarks and perspectives

In this paper we demonstrated how to generalize the quasi-classical $\bar{\partial}$-dressing method at any dispersive order. In particular, formula (66) suggests that a regular structure survives at higher orders, allowing us to define a potential encoding all information about the hierarchy at the first dispersive order, just as in the purely dispersionless case. Moreover, the paper provides us with several intriguing observations which could be subjects of future study. In particular, it would be useful to analyse the connection between the standard theory of the asymptotics approximation and the quasi-classical $\bar{\partial}$-dressing approach. This could set a relationship between the source of the obstacles in the construction of the corrections, discussed in section 6, and the caustics problem. As far as possible applications are concerned, it would be of interest to consider the KdV limit of the KP hierarchy to establish a connection with the Dubrovin–Zhang theory [3]. Another intriguing matter of study and application is associated with a possible physical interpretation of the solutions of dispersionless systems as describing integrable dynamics of interfaces [17]. In this context, one should investigate a quasiconformal dynamics described by the quasi-classical $\bar{\partial}$ problem and the corrections (37)–(39). One could analyse the singular behaviour of interfaces' evolution, by introducing, for instance, a small 'dispersive' parameter. For such a purpose, a deeper study of the singular sector order by order will also be necessary.

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